CAUCHY'S PAPER OF 1814 ON DEFINITE INTEGRALS.*

BY H. J. ETTLINGER.

Introduction. In 1814 Augustin Louis Cauchy presented before the Académie des Sciences a "memoir on definite integrals," in which appears for the first time the essence of his discoveries on residues. The memoir was first printed in 1825† with additional notes and again in 1882‡ with no change save in the matter of notation.

Although the kernel of the idea, that the integral of an analytic function of a complex variable taken along a closed path depends entirely upon the behavior of the function at points of discontinuity within the path, is contained in the paper, yet there are several reasons why the reader might notice nothing at all like this theorem. In the first place, although geometrical representation is now an essential feature of every presentation of the theory of functions, Cauchy used neither figures nor geometrical language. In the second place, the fundamental theorem, and, indeed, all the applications in this paper, concern simple integrals; but the author states the central problem as the determination of the difference in the value of an iterated integral according to the order of integration with respect to the two variables. By the use of this difference he obtains the residue, thereby obscuring the relation of the latter to a line integral. Thirdly, he refrains from using complex quantities, invariably separating an equation into its real and imaginary parts. This necessitates longer equations, more of them, clumsier notation, and a much more obscure treatment than would be the case had he used complex quantities. Cauchy himself came to appreciate this fact, for his footnotes of 1825 are devoted to the simpler complex equations from which his real ones can be readily deduced. Finally, all editions abound in misprints.

For these reasons the discoveries contained in this memoir were not appreciated even by the great mathematicians of his time. Poisson§ saw in the paper merely a means of evaluating integrals and remarked that, at least so far as the first part was concerned, no new formulæ were announced. As for the evaluation of iterated integrals by the so-called

* Presented to the Amer. Math. Soc., Sept. 2, 1919.

† "Mémoire sur les intégrales définies," Savants Étrangers, 1, p. 509, Académie des Sciences de l'Institut de France.

‡ Œuvres Complétes, I série, 1, p. 319 ff.

§ Bulletin de la Societé Philomatique (3), 1, 1814, p. 185.

"singular" integrals (which are equal to the difference between the value obtained by integrating first with respect to x and then with respect to y and the value obtained by integrating in the reverse order) he said that, though the new method was worthy of consideration, it ought not to replace the *old ones*! Lacroix and Legendre, in the official report on the paper, stated as the valuable results obtained by Cauchy: (1) the construction of a series of general formulæ for transforming and evaluating definite integrals, (2) the pointing out of the fact that the value of an iterated integral may depend on the order of integration, (3) the discovery of the cause and amount of this difference in value, (4) the derivation of new formulæ which, to be sure, might have been otherwise obtained. It seems, then, likely that the foremost mathematicians of that time failed to recognize the contributions of main importance in this paper.

To appreciate thoroughly the memoir, the following facts must be noted in addition: (1) imaginaries had no secure arithmetical basis in 1814, (2) this was the first deduction by rigorous methods of the formulæ, hitherto obtained by purely formal processes, for evaluating definite integrals, (3) while the form had not yet been cast in the ϵ-mould, which itself originated with Cauchy, nevertheless the proofs are so conceived that they correspond in substance to the standards of rigor of the present day.

Part I.

Continuous integrand. In discussing the memoir we shall frequently combine two separate real equations into one complex equation, as Cauchy did in his notes of 1825 and, very likely, in his original work. We shall also adopt the language of modern analysis for the sake of clearness and accuracy.

The first theorem proved in the memoir is, in effect, that if a function of a complex variable is analytic throughout a region of a certain type and continuous in and on the boundary, the integral of the function taken along the boundary of the region is zero.* The regions considered are mapped in a one-to-one manner and continuously, but not in general conformally, on a rectangle in the real (x, y) plane. The mapping on the complex $M + Ni$ plane is performed by taking M and N as real and continuous functions of x and y with derivatives of all orders with respect to x and y, continuous in x and y regarded as independent variables.

* For modern treatment of this theorem see Osgood, Lehrbuch der Funktionentheorie, erster Band, zweite Auflage, pp. 284–285; Pierpont, Functions of a Complex Variable, pp. 211–214; Goursat, Cours d'Analyse Mathématique, tome II, pp. 82–92. These three text-books will hereafter be referred to as O., P., G., respectively.

Let*

$$(1)\qquad f(M + Ni) = P + Qi$$

be an analytic function of $M + Ni$ in a certain region, S, of the $M + Ni$ plane, and let

$$M = \phi(x, y) \qquad \text{and} \qquad N = \psi(x, y)$$

be single-valued functions, continuous in x and y, in a rectangle, R $(0 \leqq x \leqq a,\ 0 \leqq y \leqq b)$, and on the boundary, Γ, and possessing continuous partial derivatives of all orders with respect to x and y in R and on Γ.

Furthermore, let†

$$(2)\qquad S + Ti = f(M + Ni)\,\frac{\partial(M + Ni)}{\partial x},$$

$$(3)\qquad U + Vi = f(M + Ni)\,\frac{\partial(M + Ni)}{\partial y}.$$

Differentiate (2) and (3) with regard to y and x respectively:

$$\frac{\partial S}{\partial y} + i\frac{\partial T}{\partial y} = f'(M + Ni)\,\frac{\partial(M + Ni)}{\partial x}\cdot\frac{\partial(M + Ni)}{\partial y} + f(M + Ni)\,\frac{\partial^2(M + Ni)}{\partial y\partial x},$$

$$\frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = f'(M + Ni)\,\frac{\partial(M + Ni)}{\partial y}\cdot\frac{\partial(M + Ni)}{\partial x} + f(M + Ni)\,\frac{\partial^2(M + Ni)}{\partial x\partial y}.$$

But under the conditions imposed

$$\frac{\partial^2(M + Ni)}{\partial x\partial y} = \frac{\partial^2(M + Ni)}{\partial y\partial x}\ ‡$$

or

$$\frac{\partial S}{\partial y} + \frac{i\partial T}{\partial y} = \frac{\partial U}{\partial x} + \frac{i\partial V}{\partial x}.$$

Hence

$$(4)\qquad \frac{\partial S}{\partial y} = \frac{\partial U}{\partial x} \qquad \text{and} \qquad \frac{\partial T}{\partial y} = \frac{\partial V}{\partial x}.$$

Multiplying the equations (4) by $dydx$ and integrating from $x = 0$ to $x = a$ and $y = 0$ to $y = b$, and noting further that, since the integrand is continuous, the order of integration can be reversed, we have:

* The notation of the original paper has been changed here from $P' + P''i$ to $P + Qi$ and $y = x + zi$ to $z = x + yi$.

† Hereafter $\frac{\partial(M + Ni)}{\partial x} - i\frac{\partial(M + Ni)}{\partial y}$ will be designated by $\frac{d(M + Ni)}{d(x + yi)}$.

‡ See Goursat-Hedrick, Mathematical Analysis, vol. I, p. 13.

(5) $$\int_0^a dx \int_0^b \frac{\partial S}{\partial y}\, dy = \int_0^b dy \int_0^a \frac{\partial U}{\partial x}\, dx.$$

Let $S(x, b) = S$, $S(x, 0) = s$, $U(a, y) = U$, $U(0, y) = u$; then equation (5) becomes

(6) $$\int_0^a Sdx - \int_0^a sdx = \int_0^b Udy - \int_0^b udy.$$

In a similar manner, letting $T(x, b) = T$, $T(x, 0) = t$ and $V(a, y) = V$, $V(0, y) = v$, we obtain

(7) $$\int_0^a Tdx - \int_0^a tdx = \int_0^b Vdy - \int_0^b vdy.$$

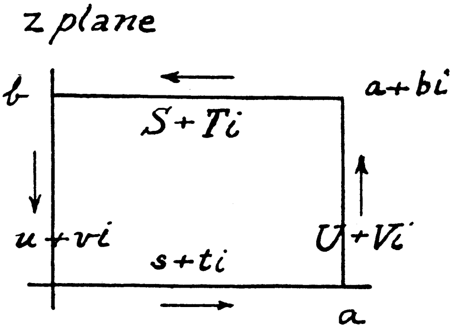

FIG. 1.

Multiplying (7) by $- i$ and (6) by $- 1$ and adding we have

$$\int_0^a (s+ ti)dx + \int_0^b (U + Vi)dy - \int_0^a (S + Ti)dx - \int_0^b (u + vi)dy = 0,$$

or

$$\int_L f(M + Ni) \frac{d(M + Ni)}{d(x + yi)}\, d(x + yi) = 0,$$

which means that around the rectangle here given in the z plane, and hence in the $M + Ni$ plane about the corresponding curve,* L_1,

(8) $$\int_L f(M + Ni)d(M + Ni) = 0.$$

Hence

FUNDAMENTAL THEOREM I: *Let $f(M + Ni)$ be an analytic function of $M + Ni$ in a certain region, S, of the $M + Ni$ plane, continuous in S, and on the boundary, L, and let $M = \phi(x, y)$ and $N = \psi(x, y)$ be single-valued continuous functions of x and y in a rectangle, R $(0 \leqq x \leqq a,\ 0 \leqq y \leqq b)$, and on the boundary, Γ, possessing continuous partial derivatives of*

* Cf. equation (3) and equations (A), footnote, Œuvres Complétes, I série, 1, p. 338. This memoir will be referred to hereafter as O.C.

all orders with respect to x and y and mapping the closed region, S, on the closed rectangle, R, in a one-to-one manner and continuously; then the integral of $f(M + Ni)d(M + Ni)$ taken around L in the positive direction vanishes, or

$$\int_L f(M + Ni)d(M + Ni) = 0.$$

In the applications Cauchy uses the real equations in S and U, T and V respectively and does not combine them as is here done. The functions used in this paper for M and N and the corresponding maps of the rectangle, R, on the $M + Ni$ plane are given in figures 2, 3, 4, and 5.

$$1°. \quad M = x, \qquad N = y.$$

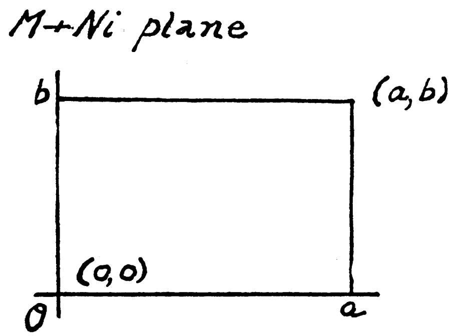

FIG. 2.

$$2°. \quad M = ax, \qquad N = xy;$$
$$a > 0.$$

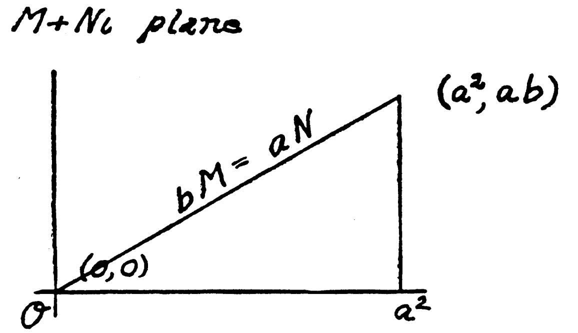

FIG. 3.

In several of the applications Cauchy allows a, the upper limit of the x-interval, to become infinite. He considered that his conclusions could be extended to this case if the function $f(M + Ni)$ approaches a limit for each value of y $(0 \leqq y \leqq b)$ when a becomes infinite and the improper integrals thus introduced converge. These conditions are, of course,

$3°.\quad M = x \cos y, \qquad N = x \sin y.$

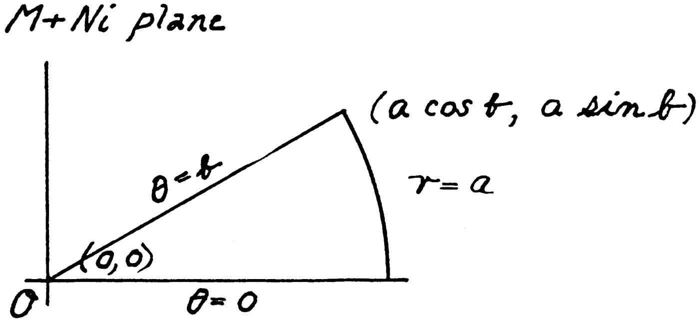

FIG. 4.

$4°.\quad M = ax^2, \qquad N = xy;$
$a > 0.$

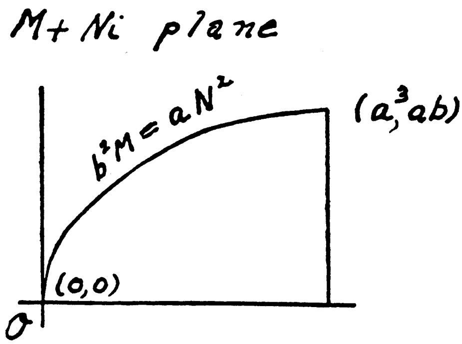

FIG. 5.

insufficient, but since all the functions $f(M + Ni)$ considered by Cauchy approach their limits uniformly in $(0 \leqq y \leqq b)$ and the integrals converge, the results may be established as correct.

The following is an example of the method of application and of the results of Part I.

Region 1°. (See Fig. 2.)

$$M = x, \qquad N = y.$$

Let $f(x + yi) = P(x, y) + Q(x, y)i$, such that $Q(x, 0) \equiv 0$, and $S + Ti = P + Qi$, $U + Vi = -Q + Pi$.

Equations (6) and (7) yield

$$(6') \qquad \int_0^a P(x, b)dx - \int_0^a P(x, 0)dx + \int_0^b Q(a, y)dy - \int_0^b Q(0, y)dy = 0,$$

$$(7') \qquad \int_0^a Q(x, b)dx - \int_0^b P(a, y)dy + \int_0^b P(0, y)dy = 0.$$

Apply these equations to

$$f(z) = e^{-z^2} \qquad \text{where} \qquad z = x + yi.^*$$

$$P(x, y) = e^{-x^2}e^{y^2}\cos 2xy, \qquad Q(x, y) = e^{-x^2}e^{y^2}\sin 2xy,$$

$$P(x, 0) = e^{-x^2}, \qquad P(0, y) = e^{y^2}, \qquad Q(0, y) = 0,$$

so that in this case equations (6′) and (7′) become respectively

$$\text{(9)} \qquad \int_0^a e^{-x^2}e^{b^2}\cos 2bx dx - e^{-a^2}\int_0^b e^{y^2}\sin 2ay dy = \int_0^a e^{-x^2}dx,$$

$$\text{(10)} \qquad -\int_0^a e^{-x^2}e^{b^2}\sin 2bx dx - e^{-a^2}\int_0^b e^{y^2}\cos 2ay dy = -\int_0^b e^{y^2}dy.$$

Now let a increase without limit. The second integral in each of the equations (9) and (10) vanishes, for

$$\left|\int_0^b e^{-a^2}e^{y^2}\sin 2ay dy\right| \leqq \int_0^b \left|\sin 2ay\ e^{-a^2}e^{y^2}dy\right.$$

$$\leqq e^{-a^2}\int_0^b e^{y^2}dy \leqq be^{b^2-a^2}, \qquad b > 0.$$

But

$$\lim_{a\to\infty} be^{b^2-a^2} = 0,$$

hence

$$\lim_{a\to\infty}\int_0^b e^{-x^2}e^{y^2}\sin 2ay dy = 0.$$

Similarly

$$\lim_{a\to\infty}\int_0^b e^{-a^2}e^{y^2}\cos 2ay dy = 0.$$

Since it can be readily shown that the other integrals converge, we are justified in writing Cauchy's equations:

$$\int_0^\infty e^{-x^2}\cos 2bx dx = e^{-b^2}\int_0^\infty e^{-x^2}dx = \frac{e^{-b^2}\sqrt{\pi}}{2},$$

$$\int_0^\infty e^{-x^2}\sin 2bx dx = e^{-b^2}\int_0^b e^{y^2}dy,$$

if we assume $\int_0^\infty e^{-x^2}dx = \sqrt{\pi}/2$.

PART II.

Integrands with poles. In the second part of the memoir Cauchy deals with the integrals of functions which are discontinuous at isolated points. In all the applications these singularities are simple poles.†

* Cf. O., p. 293, Beispiel, 4. G., p. 121, 3°.

† O.C., p. 413.

Here Cauchy obtains for the first time a formula which contains the essence of the theorem on residues. The true significance of Cauchy's method at this point is very obscure. The result is apparently stated in terms of the evaluation in two different orders of an iterated integral whose integrand has a singularity at a single point.* As a matter of fact, the iterated integral plays an unessential rôle in Part II, since all the theorems and applications are concerned with simple integrals only. Moreover no useful facts are developed concerning iterated integrals.

The exposition and criticism of Cauchy's method we lay aside for the moment and proceed to set forth a method by which the results of Part II are very simply obtained from the fundamental theorem of Part I. This method is not so very unlike Cauchy's, as will be pointed out later, and would probably be used by him were he writing in the notation of the present-day analyst. It is the method used in many modern text books on the Theory of Functions.†

FUNDAMENTAL THEOREM II: *Let $f(M + Ni)$ be an analytic function of $M + Ni$ in a certain region, S, of the $M + Ni$ plane, except for a single pole at $m + ni$, inside of S, and continuous in and on the boundary, L, of S, except at this pole. Let $M = \phi(x, y)$ and $N = \psi(x, y)$ be continuous functions of x and y in and on the boundary, Γ, of a rectangle, R $(0 \leqq x \leqq a, 0 \leqq y \leqq b)$, possessing continuous partial derivatives of all orders, mapping the closed region, S, on the closed rectangle, R, in a one-to-one manner and continuously, and such that $m = \phi(X, Y)$ and $n = \psi(X, Y)$. Let R' $(a' \leqq x \leqq a'', b' \leqq y \leqq b'')$ be any rectangle interior‡ to R and containing (X, Y) within its boundary Γ', and let L' be the curve in S corresponding to Γ'. Then the integral of $f(M + Ni) \cdot d(M + Ni)$ taken around L in the positive sense is equal to the integral of $f(M + Ni) \cdot d(M + Ni)$ taken in the positive sense around L',*

$$\int_L f(M + Ni)d(M + Ni) = \int_{L'} f(M + Ni)d(M + Ni).$$

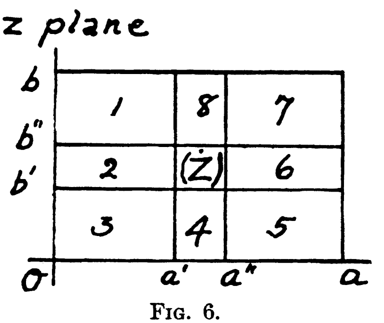

FIG. 6.

* O.C., p. 388 ff.

† O., p. 331 ff.; P., p. 206 ff.; G., p. 114 ff.

‡ I.e., $0 < a' < a'' < a$, $0 < b' < b'' < b$.

The proof follows immediately from the fundamental theorem I by application successively to the rectangles marked 1 $\cdots$ 8 in Fig. 6 and addition of the resulting equations. The equivalent of this equation in Cauchy's paper gives the first statement of his discovery on Residues.*

We proceed to derive the formulæ necessary to make the first application by evaluating $\int_{L'} f(M + Ni)d(M + Ni)$† explicitly for the case $M = x$, $N = y$, or $z = M + Ni$.

Let‡ $A + Bi = \int_{L'} f(z)dz$, and suppose:

Case 1.

$$f(z) = \frac{C}{z - Z}, \qquad \text{where} \qquad Z = X + Yi.$$

Then

$$A + Bi = \int_{L'} \frac{Cdz}{z - Z}.$$

Let

$$z - Z = re^{\phi i}, \qquad dz = ire^{\phi i}d\phi + e^{\phi i}dr.$$

Then

$$A + Bi = \int_0^{2\pi} Cid\phi + \int_{L'} \frac{dr}{r}$$

or

$$A + Bi = 2\pi iC,\S$$

since the initial and final values of r are equal.

Case 2.

$$f(z) = \phi(z) + \frac{C}{z - Z},$$

where $\phi(z)$ is analytic in R and continuous in R and on the boundary, L, and where Z is within R,

$$A + Bi = \int_{L'} \phi(z)dz + \int_{L'} \frac{C}{z - Z}dz$$

or

$$A + Bi = 2\pi iC,$$

since $\int_{L'} \phi(z)dz = 0$ by the fundamental theorem I.

Case 3. If $f(z)$ has a pole on the boundary, L, of the rectangle, R, but

* O.C., p. 381, equation (4).

† L' is identical with Γ' in this case.

‡ The notation has here been changed from $A' + A''i$ to $A + Bi$.

§ Cf. O. C., footnote, p. 412, equation (C).

not at one of the vertices, we construct R'' as in figure 7 and denote by $\bar{L}$ and L'' respectively the boundaries of the rectangles R and R'', omitting in each case the segment AB. We apply now the fundamental theorem I to the rectangles 1, 2, 3 and sum the results. In this way we find

$$\int_{\bar{L}} f(z)dz + \int_{L''} f(z)dz = 0.$$

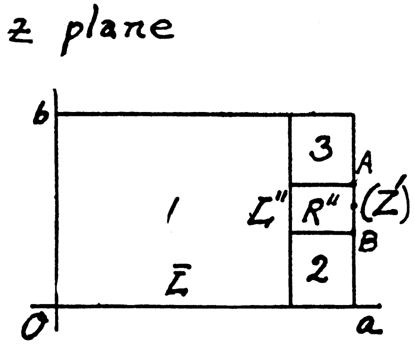

FIG. 7.

If now, in particular, we suppose $f(z) = C/(z - Z')$, we write $A + Bi = \int_{\bar{L}} f(z)dz = -\int_{L''} C/(z - Z')dz$, and $z - Z' = re^{\phi i}$; then $dz = ire^{\phi i}d\phi + e^{\phi i}dr$. Hence

$$A + Bi = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} Cid\phi + \int_{L''} C \cdot \frac{dr}{r} = \pi i C,$$

since L'' may be chosen in such a manner that the initial and final values of r are equal.

Case 4. Let $f(z) = C/(z - Z') + \phi(z)$ where $\phi(z)$ is analytic throughout R, and Z' is a point of L, not at a vertex of R. Now $\int_L \phi(z)dz = 0$ by the fundamental theorem I. Hence here also we obtain

$$A + Bi = \pi i C.$$

Case 5. In general let us consider a rational function

$$f(z) = \frac{G(z)}{F(z)} = \sum \frac{C_k}{z - Z_k} + \sum \frac{C_{k'}'}{z - Z_{k'}'} + \phi(z),$$

where $F(z)$ has only simple roots in R, and on the boundary, L, *but not at a vertex*. Then $A + Bi$ must be computed for each pole and the results summed. Hence

(11) $$A + Bi = 2\pi i \sum C_k + \pi i \sum C_{k'}'^{*}$$

where C_k is the coefficient of $1/(z - Z_k)$, Z_k an interior point of R, and where $C_{k'}'$ is the coefficient of $1/(z - Z_{k'}')$, $Z_{k'}'$ a point on the boundary not at a vertex.

Let $C_k = \lambda_k - i\mu_k$ and $C_{k'}' = \lambda_{k'}' - i\mu_{k'}'$. Then

(12) $$A = 2\pi \sum \mu_k + \pi \sum \mu_{k'}'$$

and

(13) $$B = 2\pi \sum \lambda_k + \pi \sum \lambda_{k'}'.$$

On the basis of the fundamental theorem II and equations (12) and (13) we may work out one of the examples given by Cauchy in Part II. Cauchy applies these formulæ to the rectangle bounded by $y = 0$, $y = b > 0$, $x = -a$, $x = a$, and then allows a and b to increase without limit (Fig. 8).

$$f(M + Ni) = f(z) = f(x + yi) = P + Qi,$$

where $Q(x, 0) \equiv 0$.

$$A + Bi = \int_L f(z)dz$$

$$= \int_{-a}^{a} P(x, 0)dx + \int_0^b [P(a, y) + iQ(a, y)]idy$$

$$- \int_{-a}^{a} [P(x, b) + iQ(x, b)]dx - \int_0^b [P(0, y) + iQ(0, y)]idy.$$

Separating this equation into real and imaginary parts, we have

(14) $$A = \int_{-a}^{a} P(x, 0)dx - \int_0^b Q(a, y)dy - \int_{-a}^{a} P(x, b)dx + \int_0^b Q(0, y)dy,$$

(15) $$B = \int_0^b P(a, y)dy - \int_{-a}^{a} Q(x, b)dx - \int_0^b P(0, y)dy.$$

To be able to eliminate from the formulæ all integrals except those along the axis of reals, Cauchy thinks it sufficient to take $f(z)$ to be a function such that P and Q vanish when $x = \pm \infty$, $y = \infty$. This is not at all sufficient, however, for stronger conditions are called for to insure the vanishing of the integrals in question. It is sufficient, however, if a and b increase indefinitely in a prescribed manner such as e.g.

$$\lim_{a \to \infty} \frac{b}{a} = k \neq 0$$

and that

$$\lim_{a \to \infty} \sqrt{a^2 + b^2} \max |f(x + yi)| = 0,$$

* Cf. O.C., p. 422, footnote, equation (G).

the maximum being taken for all points $\pm a + yi$ in the interval $0 \leqq y \leqq b$ and all points $x + bi$ in the interval $-a \leqq x \leqq a$, i.e., for all points on three sides of the rectangle determined by $(-a, 0)$, $(-a, b)$, (a, b), and $(a, 0)$. That is, we shall assume that, given a positive number, ϵ, arbitrarily small, we can find a positive number, X, such that

$$\sqrt{a^2 + b^2} \max |f(x + yi)| < \epsilon,$$

when $a > X$ for all points $\pm a + yi$ in the interval $0 \leqq y \leqq b$ and all points $x + bi$ in the interval $-a \leqq x \leqq a$. Then

$$\left|\int_0^b Q(\pm a, y)dy\right| \leqq \int_0^b \left|f(\pm a + yi)\right| dy$$
$$\leqq \int_0^b \frac{\epsilon}{\sqrt{a^2 + b^2}} dy < \frac{\epsilon b}{\sqrt{a^2 + b^2}} < \epsilon,$$

when $a > X$.

Hence, as a and b increase indefinitely in this prescribed manner,

$$\lim_{a,\, b \to \infty} \int_0^b Q(\pm a, y)dy = 0.$$

Similarly, it may be proved that

$$\lim_{a,\, b \to \infty} \int_{-a}^a P(x, b)dx = 0, \lim_{a,\, b \to \infty} \int_{-a}^a Q(x, b)dx = 0, \lim_{b \to \infty} \int_0^b P(0, y)dy = 0,$$
$$\lim_{b \to \infty} \int_0^b Q(0, y)dy = 0, \qquad \lim_{a,\, b \to \infty} \int_0^b P(a, y)dy = 0.$$

Moreover, if in addition $\lim_{a \to \infty} af(a) = 0$, then $\int_{-a}^a P(x, 0)dx$ converges as a increases indefinitely, for if ϵ is positive and arbitrarily small, there exists a positive number X such that $|f(x)| < \epsilon/a$ for $a > x > X$, and

$$\left|\int_X^a P(x, 0)dx\right| \leqq \int_X^a \left|f(x)\right| dx$$
$$\leqq \int_X^a \frac{\epsilon}{a} dx$$
$$\leqq \frac{\epsilon}{a}(a - X) < \epsilon$$

for $a > X$.

Formula (15) tells us that, for a function fulfilling the above conditions, $B = 0$, and the formula (14) reduces to

$$\int_{-\infty}^{\infty} pdx = A,$$

where A is taken for all the poles of $f(z)$ where $y \geqq 0$. If $f(z)$ is an even function, (16) becomes

$$A = 2\int_0^\infty p\,dx.$$

Let $f(z) = z^{2m}/(1 + z^{2n})$, where n is the greater of the two positive integers, m and n. Then $Q(x, 0) \equiv 0$ and $P(x, 0) \equiv x^{2m}/(1 + x^{2n})$, an even function. Hence

$$(17) \qquad A = 2\int_0^\infty \frac{x^{2m}}{x^{2n} + 1}\,dx.$$

We must now specify a path for $a + bi$ such that $\lim\limits_{a\to\infty} \dfrac{b}{a} = k \neq 0$ and $\lim\limits_{a\to\infty} \sqrt{a^2 + b^2}\max|f(x + yi)| = 0$ for all points $\pm a + yi$ in the interval $0 \leqq y \leqq b$ and all points $x + bi$ in the interval $-a \leqq x \leqq a$.

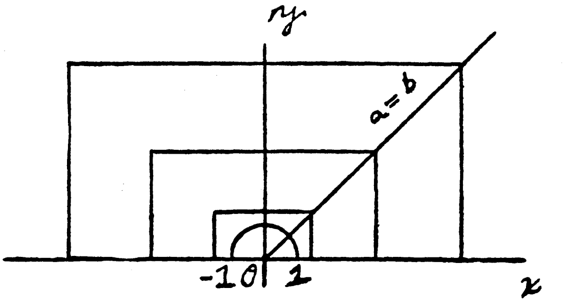

FIG. 8.

We will take $a = b$ (see Fig. 8). Then

$$\sqrt{a^2 + b^2}\max\left|f(\pm a + yi)\right| \leqq \sqrt{2}a\left|\frac{(\pm a + ai)^{2m}}{1 + (\pm a)^{2n}}\right|$$
$$\leqq \frac{\sqrt{2}\,|(\pm 1 + i)^{2m}|}{a^{2(n-m)-1}(a^{-2n} + 1)}$$
$$\leqq \frac{2^{2m+\frac{1}{2}}}{a^{-2n} + 1}\,\frac{1}{a^{2(n-m)-1}}.$$

The last expression approaches zero as a increases indefinitely.

Similarly

$$\sqrt{a^2 + b^2}\max\left|f(x + bi)\right| \leqq \sqrt{2}a\left|\frac{(a + ai)^{2m}}{1 + (ai)^{2n}}\right|$$
$$\leqq \sqrt{2}\left|\frac{(1 + i)^{2m}}{a^{-2n} + i^{2n}}\right|\frac{1}{a^{2(n-m)-1}}.$$

The last expression obviously approaches zero as a increases indefinitely. Also

$$\lim_{a\to\infty} af(a) = \lim_{a\to\infty}\frac{a^{2m+1}}{1 + a^{2n}} = 0.$$

The conditions sufficient to justify (17) are therefore satisfied.

The poles of $f(z)$ are to be found where $z^{2n}+1=0$, or

$$Z_k = e^{\pi\left(\frac{2k+1}{2n}\right)i}, \qquad k=0,1\cdots 2n-1.$$

We observe that the poles are on the unit circle and therefore certainly inside the rectangle R as soon as $a>1$.

$$\frac{z^{2m}}{1+z^{2n}} = \frac{C_0}{z-e^{\frac{\pi}{2n}i}} + \frac{C_1}{z-e^{\frac{3\pi}{2n}i}} + \cdots + \frac{C_{2n-1}}{z-e^{\frac{4n-1}{2n}\pi i}}.$$

Now

$$\begin{aligned} C_k &= \lim_{z-Z_k} \frac{z^{2m}(z-Z_k)}{1+z^{2n}} \\ &= \lim_{z-Z_k} \frac{(2m+1)z^{2m}-2mz^{2m-1}Z_k}{2nz^{2n-1}} \\ &= \frac{Z_k^{2m}}{2nZ_k^{2n-1}} = \frac{1}{2n}Z_k^{2(m-n)+1} \\ &= \frac{1}{2n}e^{\left[(2k+1)\frac{\pi i}{2n}\right](2m+1-2n)} \\ &= -\frac{1}{2n}e^{(2k+1)(2m+1)\frac{\pi i}{2n}}. \end{aligned}$$

Also

$$\begin{aligned} \sum_{k=0}^{2n-1} C_k &= -\frac{1}{2n}\sum_{k=0}^{2n-1} e^{(2k+1)\frac{2m+1}{2n}\pi i} \\ &= -\frac{1}{2n}\frac{1-e^{(2m+1)\pi i}}{1-e^{\frac{2m+1}{2n}\pi i}}e^{\frac{2m+1}{2n}\pi i} \\ &= -\frac{i}{2n}\frac{2i}{e^{\frac{2m+1}{n}\pi i}-e^{-\frac{2m+1}{n}\pi i}} = -\frac{i}{2n\sin\frac{2m+1}{2n}\pi}. \end{aligned}$$

And

$$A = 2\pi\sum_{k=0}^{2n-1}\mu_k = \frac{2\pi}{2n\sin\frac{2m+1}{2n}\pi}.$$

Hence

$$\int_0^\infty \frac{x^{2m}}{1+x^{2n}}dx = \frac{\pi}{2n\sin\frac{2m+1}{2n}\pi}.$$

Let $2m+1=\alpha$ and $2n=\beta$. Then

$$\int_0^\infty \frac{x^{\alpha-1}}{1+x^\beta}dx = \frac{\pi}{\beta\sin\frac{\alpha}{\beta}\pi},$$

a formula which Euler had obtained.

In a similar manner other formulæ are obtained by taking other functions and other regions and integrating around the corresponding rectangle.*

We return to consider the method used by Cauchy in the second part to obtain the results of the fundamental theorem II and its immediate corollaries proved above. In the first place, Cauchy adopts the "principal value" definition to remove any difficulties regarding the evaluation of simple integrals due to singularities on the path of integration.†

Suppose we have $\phi'(x)$, the derivative of a real function $\phi(x)$ of a real variable. Consider

$$\int_c^d \phi'(x)dx, \tag{19}$$

where $\phi'(x)$ has a finite or infinite discontinuity at (X) between c and d but is continuous in $c \leqq x < X$ and $X < x \leqq d$. By the "principal value" of (19) is meant

$$\lim_{h\to 0}\left[\int_c^{X-h} \phi'(x)dx + \int_{X+h}^d \phi'(x)dx\right]$$
$$= \lim_{h\to 0}[-\phi(c) + \phi(X-h) - \phi(X+h) + \phi(d)]$$
$$= \phi(d) - \phi(c) - \Delta,$$

where

$$\Delta = \lim_{h\to 0}[\phi(X+h) - \phi(X-h)].$$

According to the modern definition of convergence of an improper integral, the existence of this limit is a necessary but not a sufficient condition for the convergence of (19); Cauchy takes it as his working definition.‡ Secondly, to define the improper iterated integrals which occur, Cauchy proceeds as follows. Let $U(x, y)$ be a function which is continuous in x and y and possesses a continuous partial derivative with respect to x everywhere inside a rectangle, R $(0 \leqq x \leqq a,\ 0 \leqq y \leqq b)$, and on the boundary, L, except at the point $(0, 0)$ where it possesses a non-removable singularity but does not become infinite.§ Then‖

$$\int_0^b dy \int_0^a \frac{\partial U}{\partial x}dx = \lim_{\xi\to 0}\int_0^b dy\int_\xi^a \frac{\partial U}{\partial x}dx \tag{20}$$

where $\xi > 0$.

This definition is totally inadequate, since the simple integral obtained after a first integration does not even come under the principal value definition and may even *diverge*.

* See O., pp. 289–295; G., pp. 118–122.

† O.C., p. 402.

‡ O.C. Cf. example on p. 404, $\int_{-2}^{4} dz/z$.

§ Cf. examples given by Cauchy, O.C., p. 394 and p. 397.

‖ O.C., p. 390.

It is, however, to be noticed that these insufficient definitions do not impair the value of Cauchy's results, nor do they substantially affect the method. If the discontinuity occurs at a corner of R, the method is not applicable in general, as Cauchy's formula itself shows.* For the definition of equation (20) above is an attempt to "cut-out" the singularity. But this does not yield a convergent result for this case. We cannot, therefore, put any real content into this particular result from our modern point of view and have, therefore, excluded it in our treatment. Cauchy, himself, makes no use of this equation in any of the numerous applications of the memoir.

When the point of discontinuity occurs inside of R at (X, Y), the rectangle is divided into four parts† by the lines $x = X$, $y = Y$, and the iterated integral is separated in a manner corresponding to the double integrals over each of the four rectangles. When these four integrals are added together, we have in rather obscure form what amounts to the method which we have set forth above. The singular point has been "cut-out" by a small rectangle, $x = X - \xi$, $x = X + \xi$, $y = Y - \eta$, $y = Y + \eta$, and a method of evaluating the line integral about the small rectangle is given in the form‡

$$(21)\qquad A = \lim_{\eta \to 0} \lim_{\xi \to 0} \left[\int_y^{y+\eta} U(X + \xi, y)dy + \int_{y-\eta}^{y} U(X + \xi, y)dy \right.$$
$$\left. - \int_y^{y+\eta} U(X - \xi, y)dy - \int_{y-\eta}^{y} U(X - \xi, \xi)dy \right].$$

The bracket is substantially the real part of our equation (11). The difference between our method of evaluation of (11) and Cauchy's method of evaluating§ (21) is a striking example of the economy of the complex variable formulation.

When the point of discontinuity is on the boundary of R, the value of A is given by two terms|| of (21). In both cases the results after integration are identical with those of (12) and (13).

In the historical review of the theory of functions by Brill and Noether¶ a brief treatment of the historical importance of the memoir is given but not from a critical point of view as is here done.**

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* O.C. Cf. third equation under (20), p. 412.

† O.C., p. 396.

‡ O.C. Cf. p. 397, equation (13).

§ O.C., pp. 406–412.

|| O.C., p. 400.

¶ Jahresbericht der deutschen Mathemataken-vereinigung, vol. 3 (1894), p. 165 ff.

** The above paper has grown out of an investigation of Cauchy's work on definite integrals and residues in a Seminar course at Harvard University. Some of the early work was done with the collaboration of Dr. E. S. Allen.